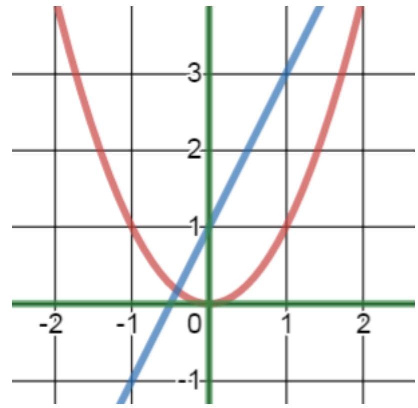
FUNCTION NOTATION AND COMPOSITION

The notation "y = *whatever*" is a fine way to write a function, but it has a major flaw. Suppose we are talking about two functions in the same problem — for example, the line $y = 2x + 1$ and the parabola $y = x^2$. I now ask you what the y-value is when $x = 1$. Is the answer 3 (from the line), or is the answer 1 (from the parabola)? It depends on which formula you chose.



If $x = 1$, what does y equal? It depends on which graph you're looking at.

❑ FUNCTION NOTATION

Not knowing which function we're referring to is unacceptable — we'd never be able to communicate this way. This is why we use a new notation for functions: We give each function a <u>different name</u>, in order to distinguish one from the other. We can name our line and parabola functions something like

$$L(x) = 2x + 1 \quad \text{and} \quad P(x) = x^2$$

[The symbol $L(x)$ is read "L **of** x" or "L **at** x".]

The L and the P are the names of the functions, and the x's in parentheses just make it clear which variable is the input. So, for example, in the function L, the input is x, and the output is $L(x)$ (that is, $2x + 1$).

Now, with this new notation, if I specify an input of $x = 1$, it's fair to ask you what $L(1)$ is, and you'll know for sure that it's 3, because I gave you both the input (the 1) and the function L. And by the same token, if I asked what $g(1)$ is, you'd be confident that it's 1. Get it??

The following are some of the math functions you will find in programming languages and spreadsheets:

```
abs
sqrt
sin
log
exp
mod
round
```

<u>EXAMPLE 1:</u> **Given the three functions**

$$\boldsymbol{f(x) = 3x - 10}$$
$$\boldsymbol{g(x) = x^2 + 1}$$
$$\boldsymbol{h(x) = \frac{1}{x}}$$

calculate each of the functional values:

A. $f(5) = 3(\mathbf{5}) - 10 = 15 - 10 = \mathbf{5}$

B. $g(5) = \mathbf{5}^2 + 1 = 25 + 1 = \mathbf{26}$

C. $h(5) = \mathbf{\frac{1}{5}}$

D. $f(10) = 3(\mathbf{10}) - 10 = 30 - 10 = \mathbf{20}$

E. $g(\sqrt{7}) = (\sqrt{\mathbf{7}})^2 + 1 = 7 + 1 = \mathbf{8}$

F. $h(0) = \frac{1}{\mathbf{0}}$, which is **undefined**

Homework

1. Let $f(x) = x^2 + 3x$

   $g(x) = 5 - x$

   $h(x) = \sqrt[3]{x+1}$

Calculate:

a. $f(7)$	b. $g(7)$	c. $f(0)$	d. $g(0)$
e. $f(10)$	f. $g(10)$	g. $f(-5)$	h. $g(-5)$
i. $h(0)$	j. $h(-28)$	k. $h(-1)$	l. $f(-1)$

❑ *COMPUTER LANGUAGE FUNCTIONS*

Let's look at how a computer language handles functions by looking at three functions: two that are built into the language, `abs` and `sqrt`, and one that we'll define ourselves.

You may have guessed that **`abs`** stands for absolute value. For example, the `abs` function produces the outputs for the given inputs:

`abs`(7) = 7 `abs`(–12) = 12 `abs`(0) = 0

As for the square root function, **`sqrt`**, a computer would calculate like this:

`sqrt`(81) = 9 `sqrt`(0) = 0 `sqrt`(–4) = *Error*

Our third example will be a *user-defined* function; this is a function that does <u>not</u> come pre-built into the language, but one that we create ourselves.

Notice that the square root of a negative number does not exist as a real number — hence the *Error* in the `sqrt` calculations above. So how can a programmer absolutely guarantee that her program will never accidentally try to take the square root of a negative number? Here's one way: Take the number's absolute value first, then feed that result (which can't be negative) into the square root function:

`sqrt`(`abs`(–9)) would produce an output of **3**.

Do you see why? Starting with the input, –9, <u>first</u> its absolute value is calculated, which results in the number 9. Second, the positive square root of the 9 is taken, with a final output of 3. Get it? What if we start with a positive number? No problem — for example,

`sqrt`(`abs`(100)) would produce an output of **10**.

The absolute value of 100 is still 100, whose positive square root is 10. Whether the input is positive, zero, or negative, our new function will always be able to take a square root, with no Error message to worry about. And this new function will even work perfectly if the input is 0.

Just for the heck of it, let's reverse the order of the functions which comprise our new user-defined function, and see how it handles an input of –25:

`abs(sqrt(–25)) = abs(`***`Error`*****!**`)`

It appears that the order in which you carry out your functions might make everything fall apart, thus defeating the purpose of a user-defined function.

And we can even name our user-defined function (the one that works, not the one that starts with `abs`), and have it stored in the computer language, available anytime we need it. For instance, we might call the new function `RealSqrt`, meaning that our new function will always result in a real number and never produce an Error message — a good thing for a programmer to have. Depending on the computer language, it might be as simple to create as this:

```
Function RealSqrt(x)
   RealSqrt(x) = sqrt(abs(x))
End Function
```

Our new function, `RealSqrt`, is actually *composed* of two functions, `abs` and `sqrt`, performed one after the other. As such, we say that our new function is the ***composition*** of the two functions. So, if you wanted to use the function with an input of –100, you could write

`RealSqrt(–100)`, and the output would be `10`.

❑ Composition of Functions

Here's the mathematical approach to the composition of functions. A function takes an input and produces exactly one output. Isn't it possible that this output might itself be the input to some other function? For example, suppose we have the two functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x - 10$$

Our starting point will be to use the input 7 and place it into the function f. This yields an output of **49**. Now, declare the 49 to be the input to the function g. The output at this point is **39**. Skipping the middle step, the original input of 7 produced — by performing f and then g — an output of **39**.

Another way to describe this "composing" of functions is the following: The starting number of 7 was put into the "squaring machine," creating the number 49. Then the 49 was put into the "subtract 10 machine," giving the final answer of **39**.

Since the function f was done first, and the function g was done second, and the starting number was 7, we can consider "g of f of 7," written

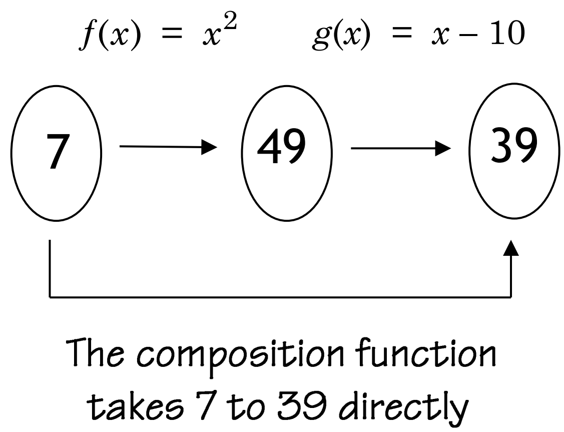
$$g\big(f(7)\big) = \mathbf{39}$$

Showing all the details, we can write

$$g\big(f(7)\big) = g\left(7^2\right) = g(49) = 49 - 10 = \mathbf{39}$$

Using the output of one function as the input to a second function is called "composing" the functions. The function that takes the 7 directly to the 39 is called the "composition" of the two functions. It's a single function which has the same effect as the two original functions, one followed by the other.

In summary,



f takes 7 to 49, and then g takes 49 to 39. We can write

$$g\big(f(7)\big) = 39$$

<u>EXAMPLE 2:</u> **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, calculate**

a. $f\big(g(-3)\big)$ **b.** $g\big(f(2)\big)$

<u>Solution:</u>

a. $f\big(g(-3)\big) = f\big(2(-3)+10\big) = f(4) = 4^3 - 4 = 64 - 4 = \boxed{60}$

b. $g\big(f(2)\big) = g\big(2^3 - 2\big) = g(6) = 2(6)+10 = \boxed{22}$

❑ *OFFICIAL NOTATION FOR THE COMPOSITION OF FUNCTIONS*

Calculating an expression of the form $f\big(g(x)\big)$ is important enough to warrant its own notation. We write

$$\boxed{(f \circ g)(x) = f\big(g(x)\big)}$$

and call $f \circ g$ the ***composition*** (or composite) of f and g. That is, given two functions f and g, we can create a third function called the composition of f and g, denoted by $f \circ g$, defined by the above equation. We can read $f\big(g(x)\big)$ as "f of g of x." Notice that the function g is done first, followed by f. Thus, function composition is done from right to left.

EXAMPLE 3: **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, find $(g \circ f)(x)$.**

Solution: By the definition of the composition of two functions,

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^3 - x) = 2(x^3 - x) + 10 =$$

$$\boxed{2x^3 - 2x + 10}$$

Function Composition is NOT Commutative

Remember the commutative property for multiplication? It says that if a and b are real numbers, then their products are equal: $ab = ba$. We'll now demonstrate that the composition of functions is a non-commutative operation. In other words, the order of the functions may make a difference in the final answer.

Let $f(x) = x^2$ and $g(x) = x + 1$. Let $x = 10$; we'll calculate

$$(f \circ g)(10) = f\big(g(10)\big) = f(10+1) = f(11) = 11^2 = \mathbf{121},$$

but

$$(g \circ f)(10) = g\big(f(10)\big) = g(10^2) = g(100) = 100+1 = \mathbf{101}.$$

Homework

2. Let $u(x) = x^2$ and $w(x) = 3x - 4$. Calculate:
   a. $u(w(3))$ b. $w(u(3))$

3. Let $f(x) = \sqrt{x}$ and $h(x) = 3x$.
   a. Calculate $(f \circ h)(1)$
   b. Explain why $(f \circ h)(-3)$ does not exist (as a real number).

4. Let $f(x) = \sqrt{x}$ and $g(x) = x + 10$.
   a. Find $(f \circ g)(-1)$ b. $(f \circ g)(-10)$ c. $(f \circ g)(-11)$

5. If $g(x) = x^2 + 1$ and $f(x) = 5x - 9$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

6. If $f(x) = 3x$ and $g(x) = \sqrt{x+1}$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

7. If $g(x) = (x + 1)^2$ and $h(x) = \sqrt{x}$, calculate formulas for $(g \circ h)(x)$ and $(h \circ g)(x)$.

8. If $f(x) = 7x - 5$, calculate a formula for $(f \circ f)(x)$.

9. If $h(x) = x^2 + x - 1$, calculate a formula for $(h \circ h)(x)$.

10. Let $f(x) = \sqrt{x+4}$. Find a formula for $(f \circ f \circ f)(n)$.

    Now calculate $(f \circ f \circ f)(437)$.

❑ TO ∞ AND BEYOND

A. Consider the following five functions:

$$f(x) = x^2 \qquad g(x) = x + 6 \qquad h(x) = \sqrt{x+17}$$

$$j = \left\{\left(\pi, \sqrt{2}\right), \left(\sqrt{2}, \pi\right)\right\} \qquad k = \{\, (a, \pi), (b, e) \,\}$$

Calculate: $(h \circ g \circ f \circ j \circ k)(a)$

B. Solve for *n*: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = 3$

C. Solve for *n*: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = w$

Solutions

1. a. $f(7) = \mathbf{7}^2 + 3(\mathbf{7}) = 49 + 21 = 70$ b. $g(7) = 5 - \mathbf{7} = -2$

c. 0 d. 5 e. 130 f. -5 g. 10

h. 10 i. 1 j. -3 k. 0 l. -2

2. a. $u(w(3)) = u(5) = 25$ b. 23

3. a. $f(h(1)) = f(3) = \sqrt{3}$ b. It would be $\sqrt{-9}$, not in $\mathbb{R}$

4. a. 3 b. 0 c. $\sqrt{-1}$, not in $\mathbb{R}$

5. $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 5(x^2+1) - 9 = 5x^2 - 4$

$(g \circ f)(x) = g(f(x)) = (5x-9)^2+1 = 25x^2-90x+82$

6. $(f \circ g)(x) = 3\sqrt{x+1}$ $\quad$ $(g \circ f)(x) = \sqrt{3x+1}$

7. $(g \circ h)(x) = (\sqrt{x}+1)^2 = x+2\sqrt{x}+1$

$(h \circ g)(x) = \sqrt{(x+1)^2} = |x+1|$

8. $(f \circ f)(x) = f(f(x)) = f(7x-5) = 7(7x-5)-5 = 49x-40$

9. x^4+2x^3-x-1

10. $\sqrt{\sqrt{\sqrt{n+4}+4}+4}$

Evaluating the function at $n = 437$ gives a result of 3.

"One child, one teacher, one book, one pen can change the world."

- *Malala Yousafzai*